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The determinants of matrices with recursive entries ☆

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Dedicated to Prof. Christian Krattenthaler on his 50th birthday

Abstract

We prove a result concerning the evaluation of determinant of a matrix, whose entries are given by certain linear inhomogeneous recurrence relations. This result generalizes previous determinants evaluations due to Krattenthaler, Neuwirth and the authors in [C. Krattenthaler, Evaluations of some determinants of matrices related to the Pascal triangle, Sém. Lothar. Combin. 47 (2002), 19, Article B47g; E. Neuwirth, Treeway Galton arrays and generalized Pascal-like determinants, Technical Report, University of Vienna, 2004, pp. 1–16; A.R. Moghaddamfar, S.M.H. Pooya, S. Navid Salehy, S. Nima Salehy, Evaluating and generalizing certain determinants, submitted for publication]. The main tool for proving our result is LU-factorization of the recursive arrays introduced in [C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005) 68–166].

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1. Introduction

Bacher in [1] considers the determinants of matrices, the coefficients of which are given by the Pascal triangle recurrence

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$$a_{i,j} = a_{i-1,j} + a_{i,j-1} \quad (i, j \geqslant 2)$$

or

$$a_{i,j} = a_{i-1,j} + a_{i,j-1} + xa_{i-1,j-1} \quad (i, j \geqslant 2)$$

with various choices for the first row $a_{1,j}$ and column $a_{i,1}$. He proved some results and stated certain conjectures about the determinants of such matrices. Some of these conjectures have been proved by Krattenthaler in [2] and by Zakrajšek and Petkovšek in [6]. In the researches done by Zakrajšek and Petkovšek, they studied the determinants of arrays with recursion

$$a_{i,j} = xa_{i,j-1} + ya_{i-1,j-1} + za_{i-1,j} \quad (i, j \geqslant 2).$$

In the above equation the coefficients x, y and z are constants, and hence all entries $a_{i,j}$ $(i, j \geqslant 2)$ are determined with one fixed recursion equation. In [5], Neuwirth notices the structure of recursive sequences of the type above more generally. Indeed, he considers the sequences $(a_{i,j})_{i,j\geqslant 1}$ satisfying the recurrence relation

$$a_{i,j} = x_{j-1}a_{i-1,j} + y_{j-1}a_{i-1,j-1} + z_{j-1}a_{i,j-1} \quad (i, j \geqslant 2)$$

for some given sequences $(x_j)_{j\geqslant 1}$, $(y_j)_{j\geqslant 1}$, $(z_j)_{j\geqslant 1}$. Here, the main difference is that he allows the recursion coefficients to depend on the column index. Neuwirth also considers the arrays characterized by the recursion relation:

$$a_{i,j} = v_{i-1}a_{i-1,j-1} + w_{i-1}a_{i-1,j} \quad (i, j \geqslant 2)$$

in which again $(v_j)_{j\geqslant 1}$, and $(w_j)_{j\geqslant 1}$, are given sequences and the recursion coefficients depend on row index. In [5], he obtained nice results presented in Theorems 5 , 6 and Corollaries 7, 8 (see also Theorems 44 and 45 in [3]).

Here we attempt to evaluate the determinants of similar matrices and generalize some of the results obtained. We first define some matrices whose entries are determined as recursive and associated to some sequences.

Definition. Let $\alpha = (\alpha_i)_{i\geqslant 1}$, $\beta = (\beta_i)_{i\geqslant 1}$, $\gamma = (\gamma_i)_{i\geqslant 1}$, $\mu = (\mu_i)_{i\geqslant 1}$, $\nu = (\nu_i)_{i\geqslant 1}$, $\delta = (\delta_i)_{i\geqslant 1}$ and $\lambda = (\lambda_i)_{i\geqslant 0}$ be given sequences. Let

$$\begin{cases} \Phi(i,j) = \delta_{i-1}\gamma_{j-1} - \nu_{i-1}\mu_{j-1} & \text{for } i, j \geqslant 2, \\ \Psi(i,j) = \delta_{i-1}\lambda_{j-1} + \nu_{i-1} & \text{for } i \geqslant 2,\ j \geqslant 1, \\ \Omega(i,j) = [\alpha_i - \Psi(i,1)\alpha_{i-1}](\beta_j - \mu_{j-1}\beta_{j-1}) & \text{for } i, j \geqslant 2. \end{cases} \tag{1}$$

We define a matrix $A = (a_{i,j})$ of order n by setting $a_{1,i} = \alpha_1\beta_i$, and $a_{i,1} = \beta_1\alpha_i$ for $1 \leqslant i \leqslant n$ and

$$a_{i,j} = \mu_{j-1}a_{i,j-1} + \Phi(i,j)a_{i-1,j-1} + \Psi(i,j)a_{i-1,j} + \Omega(i,j)$$

for $2 \leqslant i, j \leqslant n$.

It is worth mentioning that the above recurrence is inhomogeneous. The principal objective of this research is to prove the following main theorem.

Main theorem. *The determinant of A defined as above is given by*

$$\det(A) = \alpha_1^n \prod_{l=1}^{n-1} \delta_l^{n-l} \cdot \prod_{p=1}^{n} \left\{ \sum_{k=1}^{p-1} \left[\beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2} (\lambda_{p-1} - \lambda_r) \right. \right.$$

$$\left.\left.\times \prod_{s=k+1}^{p-1}(\gamma_s+\mu_s\lambda_{p-1})\right] + \beta_p \prod_{t=0}^{p-2}(\lambda_{p-1}-\lambda_t)\right\}.$$

Note that, when $m > n$, we assume that

$$\sum_{i=m}^{n} a_i = 0 \quad \text{and} \quad \prod_{j=m}^{n} b_j = 1.$$

Moreover, in the expressions like

$$\prod_{i=k}^{l} a_i \prod_{j=s}^{t} b_j \quad \left(\text{resp. } \sum_{i=k}^{l} a_i \prod_{j=s}^{t} b_j \text{ or } \prod_{i=k}^{l} a_i \sum_{j=s}^{t} b_j\right), \tag{$*$}$$

if $k > l$ and the scope is the rest of the expression, then $(*) = 1$ (resp. 0 or 1). On the other hand, if the expression means $\left(\prod_{i=k}^{l} a_i\right)\left(\prod_{j=s}^{t} b_j\right)\left(\text{resp. } \left(\sum_{i=k}^{l} a_i\right)\left(\prod_{j=s}^{t} b_j\right) \text{ or } \left(\prod_{i=k}^{l} a_i\right)\left(\sum_{j=s}^{t} b_j\right)\right)$, then $(*) = \prod_{j=s}^{t} b_j$ (resp. $\prod_{j=s}^{t} b_j$ or $\sum_{j=s}^{t} b_j$).

As some consequences of the main theorem we have the following results.

Theorem 1 (Theorem A in [4]). *In the Main Theorem, if we take* $\beta_i = \prod_{k=1}^{i-1} \mu_k$, $\nu_i = 0$ *and* $\delta_i = 1$ *for* $(i \geqslant 1)$, *then we have*

$$a_{i,j} = \mu_{j-1}a_{i,j-1} + \gamma_{j-1}a_{i-1,j-1} + \lambda_{j-1}a_{i-1,j} \quad (2 \leqslant i, j \leqslant n)$$

and we obtain

$$\det(A) = \alpha_1^n \prod_{i=2}^{n}\prod_{k=1}^{i-1}(\mu_k\lambda_{i-1}+\gamma_k).$$

Proof. For convenience we put

$$A_k := \left(\prod_{m=1}^{k-1}\mu_m\right)(\gamma_k+\mu_k\lambda_{k-1})\prod_{r=0}^{k-2}(\lambda_{p-1}-\lambda_r)\prod_{s=k+1}^{p-1}(\gamma_s+\mu_s\lambda_{p-1}).$$

When $1 \leqslant k \leqslant p-1$, one can easily show that

$$\begin{aligned} A_k &+ \left(\prod_{m=1}^{k}\mu_m\right)\prod_{r=0}^{k-1}(\lambda_{p-1}-\lambda_r)\prod_{s=k+1}^{p-1}(\gamma_s+\mu_s\lambda_{p-1}) \\ &= \left(\prod_{m=1}^{k-1}\mu_m\right)\prod_{r=0}^{k-2}(\lambda_{p-1}-\lambda_r)\prod_{s=k}^{p-1}(\gamma_s+\mu_s\lambda_{p-1}). \end{aligned} \tag{2}$$

Now, from main theorem we obtain

$$\begin{aligned} \det(A) &= \alpha_1^n \prod_{p=1}^{n}\left\{\sum_{k=1}^{p-1}A_k + \left(\prod_{m=1}^{p-1}\mu_m\right)\prod_{t=0}^{p-2}(\lambda_{p-1}-\lambda_t)\right\} \\ &= \alpha_1^n \prod_{p=2}^{n}\left\{\sum_{k=1}^{p-1}A_k + \left(\prod_{m=1}^{p-1}\mu_m\right)\prod_{t=0}^{p-2}(\lambda_{p-1}-\lambda_t)\right\} \end{aligned}$$

$$= \alpha_1^n \prod_{p=2}^{n} \left\{ \sum_{k=1}^{p-2} A_k + \left(\prod_{m=1}^{p-2} \mu_m \right) \prod_{t=0}^{p-3} (\lambda_{p-1} - \lambda_t)(\gamma_{p-1} + \mu_{p-1}\lambda_{p-1}) \right\}$$
(by Eq. (2))
$$= \alpha_1^n \prod_{p=2}^{n} \left\{ \sum_{k=1}^{p-3} A_k + \left(\prod_{m=1}^{p-3} \mu_m \right) \prod_{t=0}^{p-4} (\lambda_{p-1} - \lambda_t) \prod_{s=p-2}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \right\}$$
(by Eq. (2))

$\vdots$

$$= \alpha_1^n \prod_{p=2}^{n} \left\{ \sum_{k=1}^{2} A_k + \left(\prod_{m=1}^{2} \mu_m \right) \prod_{t=0}^{1} (\lambda_{p-1} - \lambda_t) \prod_{s=3}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \right\}$$
(by Eq. (2))
$$= \alpha_1^n \prod_{p=2}^{n} \left\{ A_1 + \mu_1(\lambda_{p-1} - \lambda_0) \prod_{s=2}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \right\}$$
(by Eq. (2))
$$= \alpha_1^n \prod_{p=2}^{n} \prod_{s=1}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \quad \text{(by Eq. (2))}$$

as desired. □

Remark 1. In Theorem 1, if we put $\alpha_1 = 1$, then we obtain

$$\det(A) = \prod_{i=2}^{n} \prod_{k=1}^{i-1} (\mu_k\lambda_{i-1} + \gamma_k),$$

which is Theorem 44 in [3].

Theorem 2 (Theorem B in [4]). *In the main theorem, if we take $\alpha_i = \lambda_0^{i-1}$, $\nu_i = 0$ and $\delta_i = 1$ for $i \geqslant 1$, then we have*

$$a_{i,j} = \mu_{j-1}a_{i,j-1} + \gamma_{j-1}a_{i-1,j-1} + \lambda_{j-1}a_{i-1,j} \quad (2 \leqslant i, j \leqslant n)$$

and we obtain

$$\det(A) = \prod_{p=1}^{n} \left\{ \sum_{k=1}^{p-1} \left[\beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2} (\lambda_{p-1} - \lambda_r) \prod_{s=k+1}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \right] + \beta_p \prod_{t=0}^{p-2} (\lambda_{p-1} - \lambda_t) \right\}.$$

Proof. Obvious. □

Remark 2. In Theorem 2, if we put $\beta_1 = 1$ and $\lambda_i = \lambda_0$, then we have

$$\det(A) = \prod_{i=1}^{n-1} (\lambda_0\mu_i + \gamma_i)^{n-i},$$

which is Theorem 45 in [3].

Remark 3. In Theorem 2, if we put $\beta_i = 1$ and $\gamma_i = \mu_i = 0$, then we have

$$\det(A) = \prod_{p=1}^{n} \prod_{t=0}^{p-2} (\lambda_{p-1} - \lambda_t),$$

which is the Vandermonde determinant. For instance, in the case $n = 5$ the matrix A is as follows:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_0^2 & \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_0^3 & \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_0^4 & \lambda_1^4 & \lambda_2^4 & \lambda_3^4 & \lambda_4^4 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & \lambda_0^3 & \lambda_0^4 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 & \lambda_3^4 \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \lambda_4^4 \end{bmatrix}^{\mathrm{T}}.$$

Remark 4. In Theorem 2, if we put $\beta_i = \rho^{i-1}$, $\gamma_i = x$, $\mu_i = 1$ and $\lambda_i = 1$ for $i \geqslant 1$, then we have

$$a_{i,j} = a_{i,j-1} + xa_{i-1,j-1} + a_{i-1,j} \quad (2 \leqslant i, j \leqslant n)$$

and we deduce that

$$\det(A) = (1 + x)^{\binom{n-1}{2}} (x + \rho + \lambda_0 - \rho\lambda_0)^{n-1}.$$

Note that this is Theorem 41 in [3]. Indeed, from Theorem 2 we deduce that

$$\begin{aligned}
\det(A) &= \prod_{p=1}^{n} \left\{ \sum_{k=1}^{p-1} \left[\rho^{k-1}(x + \lambda_{k-1}) \prod_{r=0}^{k-2} (\lambda_{p-1} - \lambda_r) \prod_{s=k+1}^{p-1} (x + \lambda_{p-1}) \right] \right. \\
&\quad \left. + \rho^{p-1} \prod_{t=0}^{p-2} (\lambda_{p-1} - \lambda_t) \right\} \\
&= \prod_{p=2}^{n} \left\{ \sum_{k=1}^{p-1} \left[\rho^{k-1}(x + \lambda_{k-1}) \prod_{r=0}^{k-2} (\lambda_{p-1} - \lambda_r) \prod_{s=k+1}^{p-1} (x + \lambda_{p-1}) \right] \right. \\
&\quad \left. + \rho^{p-1} \prod_{t=0}^{p-2} (\lambda_{p-1} - \lambda_t) \right\} \\
&= (x + \rho + \lambda_0 - \rho\lambda_0) \prod_{p=3}^{n} \sum_{k=1}^{2} \left[\rho^{k-1}(x + \lambda_{k-1}) \prod_{r=0}^{k-2} (\lambda_{p-1} - \lambda_r) \prod_{s=k+1}^{p-1} (x + 1) \right]
\end{aligned}$$

$$= (x+\rho+\lambda_0-\rho\lambda_0)\prod_{p=3}^{n}[(x+\lambda_0)(x+1)^{p-2}+\rho(1-\lambda_0)(x+1)^{p-2}]$$

$$= (x+\rho+\lambda_0-\rho\lambda_0)\prod_{p=3}^{n}[(x+1)^{p-2}(x+\rho+\lambda_0-\rho\lambda_0)]$$

$$= (1+x)^{\binom{n-1}{2}}(x+\rho+\lambda_0-\rho\lambda_0)^{n-1}.$$

Theorem 3 (Theorem C in [4]). *In the main theorem, if we take $\lambda_i = 0$ for $i \geqslant 0$, then we have*

$$\begin{aligned} a_{i,j} &= \mu_{j-1}a_{i,j-1} + \Phi(i,j)a_{i-1,j-1} + \nu_{i-1}a_{i-1,j} \\ &\quad + (\alpha_i - \nu_{i-1}\alpha_{i-1})(\beta_j - \mu_{j-1}\beta_{j-1}) \quad (2 \leqslant i, j \leqslant n) \end{aligned}$$

and we obtain

$$\det(A) = (\alpha_1\beta_1)^n \prod_{k=1}^{n-1}(\delta_k\gamma_k)^{n-k}.$$

Proof. From main theorem we obtain

$$\begin{aligned} \det(A) &= \left(\alpha_1^n \prod_{k=1}^{n-1}\delta_k^{n-k}\right)\beta_1\prod_{p=2}^{n}\left(\beta_1\gamma_1\prod_{s=2}^{p-1}\gamma_s\right) \\ &= \left(\alpha_1^n \prod_{k=1}^{n-1}\delta_k^{n-k}\right)\beta_1^n\gamma_1^{n-1}\prod_{p=3}^{n}\prod_{s=2}^{p-1}\gamma_s \\ &= \left(\alpha_1^n \prod_{k=1}^{n-1}\delta_k^{n-k}\right)\beta_1^n\gamma_1^{n-1}\prod_{r=2}^{n-1}\gamma_r^{n-r} \\ &= (\alpha_1\beta_1)^n\prod_{k=1}^{n-1}(\delta_k\gamma_k)^{n-k} \end{aligned}$$

as desired. □

Remark 5. In Theorem 3, if we put $\beta_1 = 1, \alpha_i = c^{i-1}, \delta_i = 1, \nu_i = c$ and if $\phi = (\phi_i)_{i\geqslant 1}$ is a sequence with $\phi_i := \gamma_i - c\mu_i$ for $i \geqslant 1$, then we have

$$a_{i,j} = \mu_{j-1}a_{i,j-1} + \phi_{j-1}a_{i-1,j-1} + ca_{i-1,j} \quad (2 \leqslant i, j \leqslant n)$$

and we get

$$\det(A) = \prod_{k=1}^{n-1}(\phi_k + c\mu_k)^{n-k},$$

which is Theorem 45 in [3].

We can summarize the above results in Table 1.

Table 1

α_i	β_i	γ_i	μ_i	ν_i	δ_i	λ_i	$\det(A)$	References
α_i	$\prod_{k=1}^{i-1}\mu_k$	γ_i	μ_i	0	1	λ_i	$\alpha_1^n\prod_{i=2}^{n}\prod_{k=1}^{i-1}(\mu_k\lambda_{i-1}+\gamma_k)$	Theorem A in [4]
$\alpha_1=1$	$\prod_{k=1}^{i-1}\mu_k$	γ_i	μ_i	0	1	λ_i	$\prod_{i=2}^{n}\prod_{k=1}^{i-1}(\mu_k\lambda_{i-1}+\gamma_k)$	Theorem 44 in [3]
λ_0^{i-1}	β_i	γ_i	μ_i	0	1	λ_i	D^{a}	Theorem B in [4]
λ_0^{i-1}	$\beta_1=1$	γ_i	μ_i	0	1	λ_0	$\prod_{i=1}^{n-1}(\lambda_0\mu_i+\gamma_i)^{n-i}$	Theorem 45 in [3]
λ_0^{i-1}	1	0	0	0	1	λ_i	$\prod_{p=1}^{n}\prod_{t=0}^{p-2}(\lambda_{p-1}-\lambda_t)$	Vandermonde determinant
λ_0^{i-1}	ρ^{i-1}	x	1	0	1	1 $(i \geqslant 1)$	$(1+x)^{\binom{n-1}{2}}\Delta^{n-1}$ with $\Delta=x+\rho+\lambda_0-\rho\lambda_0$	Theorem 41 in [3]
α_i	β_i	γ_i	μ_i	ν_i	δ_i	0	$(\alpha_1\beta_1)^n\prod_{k=1}^{n-1}(\delta_k\gamma_k)^{n-k}$	Theorem C in [4]
c^{i-1}	$\beta_1=1$	$\phi_i+c\mu_i$	μ_i	c	1	0	$\prod_{k=1}^{n-1}(\phi_k+c\mu_k)^{n-k}$	Theorem 45 in [3]

[a] $D=\prod_{p=1}^{n}\left\{\sum_{k=1}^{p-1}[\beta_k(\gamma_k+\mu_k\lambda_{k-1})\prod_{r=0}^{k-2}(\lambda_{p-1}-\lambda_r)\prod_{s=k+1}^{p-1}(\gamma_s+\mu_s\lambda_{p-1})]+\beta_p\prod_{t=0}^{p-2}(\lambda_{p-1}-\lambda_t)\right\}.$

2. Proving the main theorem

We start with the following lemma.

Lemma 1. *Let m and j be integers with $1 \leqslant m \leqslant j-2$. Let $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_j$ be indeterminates. Then, for any positive integer $0 \leqslant d \leqslant j-m-1$, there holds*

$$\sum_{l=0}^{j-m}\left\{(-1)^l\lambda_{j-l-1}^d\prod_{\substack{m-1\leqslant e<f\leqslant j-1\\ e,f\neq j-l-1}}(\lambda_f-\lambda_e)\right\}=0.$$

We omit the straightforward proof.

Lemma 2. *Let $\beta=(\beta_i)_{i\geqslant 1}$, $\gamma=(\gamma_i)_{i\geqslant 1}$, $\mu=(\mu_i)_{i\geqslant 1}$ and $\lambda=(\lambda_i)_{i\geqslant 0}$ be given sequences. Let $U=(U_{i,j})_{1\leqslant i,j\leqslant n}$ be a matrix of order n given by the recurrence*

$$\begin{aligned}U_{i,j}&=\mu_{j-1}U_{i,j-1}+(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})U_{i-1,j-1}\\&\quad+(\lambda_{j-1}-\lambda_{i-2})U_{i-1,j}\quad\text{for } 2\leqslant i,j\leqslant n\end{aligned}\tag{3}$$

and the initial conditions $U_{1,1}=\beta_1$, $U_{i,1}=0$ and $U_{1,i}=\beta_i$ $(2\leqslant i\leqslant n)$. Then, the (i,j)-entry $U_{i,j}$ of U is also given by the formula

$$U_{i,j}=\sum_{l=0}^{j-i}\Omega(j,l)\Gamma(i,j,l)\Lambda(j-l)\quad(i,j\geqslant 1),\tag{4}$$

where

$$\Omega(j,l)=\prod_{s=j-l}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1}),$$

$$\Gamma(i,j,l)=\prod_{\substack{0\leqslant t\leqslant j-i\\ t\neq j-i-l}}(\lambda_{j-l-1}-\lambda_{i+t-1})^{-1},$$

$$\Lambda(j-l)=\sum_{k=1}^{j-l-1}\left\{\beta_k(\gamma_k+\mu_k\lambda_{k-1})\prod_{r=0}^{k-2}(\lambda_{j-l-1}-\lambda_r)\prod_{s=k+1}^{j-l-1}(\gamma_s+\mu_s\lambda_{j-l-1})\right\}$$
$$+\beta_{j-l}\prod_{t=0}^{j-l-2}(\lambda_{j-l-1}-\lambda_t).$$

Proof. We proceed by induction on $i+j$. If $i+j=2$, then $i=j=1$ and the result is obvious. Therefore, we assume that $i+j>2$. Note that, if $j<i$, we have nothing to prove. Now, we consider two cases separately.

First, we assume that $i=1$. In this case, we have to prove $U_{1,j}=\beta_j$. To prove this, it is enough to show that

$$\text{the coefficient of } \beta_m \text{ in } U_{1,j}=\begin{cases}1, & m=j,\\ 0, & 1\leqslant m\leqslant j-1.\end{cases}$$

Indeed, the coefficient of β_j in $U_{1,j}$ is equal to

$$\prod_{t=0}^{j-2}(\lambda_{j-1}-\lambda_t)^{-1}\prod_{t=0}^{j-2}(\lambda_{j-1}-\lambda_t)=1.$$

The coefficient of β_{j-1} in $U_{1,j}$ is also equal to

$$(\gamma_{j-1}+\mu_{j-1}\lambda_{j-2})\left[(\lambda_{j-1}-\lambda_{j-2})^{-1}+(\lambda_{j-2}-\lambda_{j-1})^{-1}\right]=0.$$

In what follows, we may, therefore, assume that $0\leqslant m\leqslant j-2$. Now, the coefficient of β_m in $U_{1,j}$ is equal to

$$(\gamma_m+\mu_m\lambda_{m-1})\sum_{l=0}^{j-m}\left\{\prod_{s=j-l}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1})\prod_{\substack{0\leqslant t\leqslant j-1\\ t\neq j-l-1}}(\lambda_{j-l-1}-\lambda_t)^{-1}\right.$$
$$\left.\times\prod_{r=0}^{m-2}(\lambda_{j-l-1}-\lambda_r)\prod_{s=m+1}^{j-l-1}(\gamma_s+\mu_s\lambda_{j-l-1})\right\}$$

or equivalently, it is equal to

$$(\gamma_m+\mu_m\lambda_{m-1})\sum_{l=0}^{j-m}\left\{\prod_{s=m+1}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1})\prod_{\substack{m-1\leqslant t\leqslant j-1\\ t\neq j-l-1}}(\lambda_{j-l-1}-\lambda_t)^{-1}\right\}.$$

Here, we show that

$$\sum_{l=0}^{j-m}\left\{\prod_{s=m+1}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1})\prod_{\substack{m-1\leqslant t\leqslant j-1\\ t\neq j-l-1}}(\lambda_{j-l-1}-\lambda_t)^{-1}\right\}=0$$

or equivalently

$$\prod_{m-1\leqslant e<f\leqslant j-1}(\lambda_f-\lambda_e)\times\sum_{l=0}^{j-m}\left\{\prod_{s=m+1}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1})\prod_{\substack{m-1\leqslant t\leqslant j-1\\ t\neq j-l-1}}(\lambda_{j-l-1}-\lambda_t)^{-1}\right\}$$

$$=\sum_{l=0}^{j-m}\left\{(-1)^l\prod_{s=m+1}^{j-1}(\gamma_s+\mu_s\lambda_{j-l-1})\prod_{\substack{m-1\leqslant e<f\leqslant j-1\\ e,f\neq j-l-1}}(\lambda_f-\lambda_e)\right\}=0.$$

To do this, we see that the coefficient of the term $\prod_{s=m+1}^{j-1}\gamma_s^{\delta_s}\mu_s^{\epsilon_s}$ where $\delta_s,\epsilon_s\in\{0,1\}$ and $\delta_s+\epsilon_s=1$, is equal to

$$\sum_{l=0}^{j-m}\left\{(-1)^l\lambda_{j-l-1}^d\prod_{\substack{m-1\leqslant e<f\leqslant j-1\\ e,f\neq j-l-1}}(\lambda_f-\lambda_e)\right\},$$

where $d=|\{s\colon\epsilon_s=1\}|$. Note that $0\leqslant d\leqslant j-m-1$. Now, Lemma 1 shows this coefficient is equal to zero.

From now on, we consider the case $2\leqslant i\leqslant j$. By induction hypothesis, we can rewrite Eq. (3) in the following way:

$$\begin{aligned}U_{i,j}&=\mu_{j-1}U_{i,j-1}+(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})U_{i-1,j-1}+(\lambda_{j-1}-\lambda_{i-2})U_{i-1,j}\\&=\mu_{j-1}\sum_{l=0}^{j-i-1}\Omega(j-1,l)\Gamma(i,j-1,l)\Lambda(j-1-l)\\&\quad+(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})\sum_{l=0}^{j-i}\Omega(j-1,l)\Gamma(i-1,j-1,l)\Lambda(j-1-l)\\&\quad+(\lambda_{j-1}-\lambda_{i-2})\sum_{l=0}^{j-i+1}\Omega(j,l)\Gamma(i-1,j,l)\Lambda(j-l).\end{aligned}\tag{5}$$

Now, it is enough to show that the corresponding coefficient of $\Lambda(j-l)$ for $l=0,1,\ldots,j-i+1$, in Eqs. (4) and (5) are the same.

First, we assume that $l=0$. In this case, the coefficient of $\Lambda(j)$ on the right-hand side of Eq. (5) is equal to

$$\begin{aligned}(\lambda_{j-1}-\lambda_{i-2})\prod_{t=0}^{j-i}(\lambda_{j-1}-\lambda_{i-2+t})^{-1}&=\prod_{t=1}^{j-i}(\lambda_{j-1}-\lambda_{i-2+t})^{-1}\\&=\prod_{t=0}^{j-i-1}(\lambda_{j-1}-\lambda_{i-1+t})^{-1},\end{aligned}$$

which is equal to the coefficient of $\Lambda(j)$ on the right-hand side of Eq. (4).

Now, we assume that $1 \leqslant l \leqslant j-i$. Here, the coefficient of $\Lambda(j-l)$ on the right-hand side of Eq. (5) is equal to

$$\begin{aligned}
&\mu_{j-1}\Omega(j-1,l-1)\Gamma(i,j-1,l-1)+(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})\Omega(j-1,l-1)\\
&\quad\times\Gamma(i-1,j-1,l-1)+(\lambda_{j-1}-\lambda_{i-2})\Omega(j,l)\Gamma(i-1,j,l)\\
&\quad\quad=\mu_{j-1}\Omega(j-1,l-1)\Gamma(i,j-1,l-1)+\Omega(j-1,l-1)\Gamma(i-1,j-1,l-1)\\
&\quad\quad\quad\times\left[(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})+\frac{(\lambda_{j-1}-\lambda_{i-2})(\gamma_{j-1}+\mu_{j-1}\lambda_{j-l-1})}{\lambda_{j-l-1}-\lambda_{j-1}}\right]\\
&\quad\quad=\mu_{j-1}\Omega(j-1,l-1)\Gamma(i,j-1,l-1)+\Omega(j-1,l-1)\Gamma(i-1,j-1,l-1)\\
&\quad\quad\quad\times\left[\frac{(\gamma_{j-1}+\mu_{j-1}\lambda_{j-1})(\lambda_{j-l-1}-\lambda_{i-2})}{\lambda_{j-l-1}-\lambda_{j-1}}\right]\\
&\quad\quad=\mu_{j-1}\Omega(j-1,l-1)\Gamma(i,j-1,l-1)+(\gamma_{j-1}+\mu_{j-1}\lambda_{j-1})\Omega(j-1,l-1)\\
&\quad\quad\quad\times\prod_{\substack{1\leqslant t\leqslant j-i+1\\ t\neq j-i-l+1}}(\lambda_{j-l-1}-\lambda_{i-2+t})^{-1}\\
&\quad\quad=\mu_{j-1}\Omega(j-1,l-1)\Gamma(i,j-1,l-1)+(\gamma_{j-1}+\mu_{j-1}\lambda_{j-1})\Omega(j-1,l-1)\\
&\quad\quad\quad\times\prod_{\substack{0\leqslant t\leqslant j-i\\ t\neq j-i-l}}(\lambda_{j-l-1}-\lambda_{i-1+t})^{-1}\\
&\quad\quad=\Omega(j-1,l-1)\Gamma(i,j-1,l-1)\left[\mu_{j-1}+\frac{\gamma_{j-1}+\mu_{j-1}\lambda_{j-1}}{\lambda_{j-l-1}-\lambda_{j-1}}\right]\\
&\quad\quad=\Omega(j,l)\Gamma(i,j,l),
\end{aligned}$$

which is equal to the coefficient of $\Lambda(j-l)$ on the right-hand side of Eq. (4). For the case $l=j-i+1$, it is easy to show that the coefficient of $\Lambda(j-l)$ in Eqs. (4) and (5) is zero. This completes the proof of the lemma. □

Now, we are ready for the main theorem.

Proof of main theorem. We apply the LU-factorization method (see [3]) . We claim that

$$A = L \cdot U,$$

where $L=(L_{i,j})_{1\leqslant i,j\leqslant n}$ with $L_{1,1}=\alpha_1$, $L_{1,i}=0$ and $L_{i,1}=\alpha_i$ $(2\leqslant i\leqslant n)$ and

$$L_{i,j}=\delta_{i-1}L_{i-1,j-1}+\Psi(i,j)L_{i-1,j} \quad (2\leqslant i,j\leqslant n) \tag{6}$$

and where $U=(U_{i,j})_{1\leqslant i,j\leqslant n}$ with $U_{1,1}=\beta_1$, $U_{i,1}=0$ and $U_{1,i}=\beta_i$ $(2\leqslant i\leqslant n)$ and

$$\begin{aligned}
U_{i,j}&=\mu_{j-1}U_{i,j-1}+(\gamma_{j-1}+\mu_{j-1}\lambda_{i-2})U_{i-1,j-1}\\
&\quad+(\lambda_{j-1}-\lambda_{i-2})U_{i-1,j} \quad (2\leqslant i,j\leqslant n).
\end{aligned}$$

The matrix L is a lower triangular one with diagonal entries

$$L_{1,1}=\alpha_1,\ L_{2,2}=\alpha_1\delta_1,\ L_{3,3}=\alpha_1\delta_1\delta_2,\ldots,\ L_{i,i}=\alpha_1\prod_{k=1}^{i-1}\delta_k,\ \ldots,\ L_{n,n}=\alpha_1\prod_{k=1}^{n-1}\delta_k,$$

whereas the matrix U is an upper triangular one with diagonal entries

$$U_{j,j} = \sum_{k=1}^{j-1} \left\{ \beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2}(\lambda_{j-1} - \lambda_r) \prod_{s=k+1}^{j-1} (\gamma_s + \mu_s\lambda_{j-1}) \right\} + \beta_j \prod_{t=0}^{j-2}(\lambda_{j-1} - \lambda_t)$$

for $j = 1, 2, \ldots, n$, by Lemma 2, that is

$$\begin{aligned}
U_{1,1} &= \beta_1, \\
U_{2,2} &= \beta_1(\gamma_1 + \mu_1\lambda_0) + \beta_2(\lambda_1 - \lambda_0), \\
U_{3,3} &= \beta_1(\gamma_1 + \mu_1\lambda_0)(\gamma_2 + \mu_2\lambda_2) + \beta_2(\gamma_2 + \mu_2\lambda_1)(\lambda_2 - \lambda_0) + \beta_3(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1), \\
&\vdots \\
U_{i,i} &= \sum_{k=1}^{i-1} \left[\beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2}(\lambda_{i-1} - \lambda_r) \prod_{s=k+1}^{i-1} (\gamma_s + \mu_s\lambda_{i-1}) \right] \\
&\quad + \beta_i \prod_{t=0}^{i-2}(\lambda_{i-1} - \lambda_t), \\
&\vdots \\
U_{n,n} &= \sum_{k=1}^{n-1} \left[\beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2}(\lambda_{n-1} - \lambda_r) \prod_{s=k+1}^{n-1} (\gamma_s + \mu_s\lambda_{n-1}) \right] \\
&\quad + \beta_n \prod_{t=0}^{n-2}(\lambda_{n-1} - \lambda_t).
\end{aligned}$$

It is obvious that the claimed factorization of A immediately implies that

$$\det(A) = \alpha_1^n \prod_{l=1}^{n-1} \delta_l^{n-l} \cdot \prod_{p=1}^{n} \left\{ \sum_{k=1}^{p-1} \left[\beta_k(\gamma_k + \mu_k\lambda_{k-1}) \prod_{r=0}^{k-2}(\lambda_{p-1} - \lambda_r) \times \prod_{s=k+1}^{p-1} (\gamma_s + \mu_s\lambda_{p-1}) \right] + \beta_p \prod_{t=0}^{p-2}(\lambda_{p-1} - \lambda_t) \right\}$$

as desired.

In order to prove the factorization stated here, i.e. $A = L \cdot U$, we calculate the (i, j)-entry of $L \cdot U$, that is, $(L \cdot U)_{i,j} = \sum_{k=1}^{j} L_{i,k}U_{k,j}$, and indeed we must demonstrate

$$\begin{aligned}
R_1(L \cdot U) &= R_1(A) = (\alpha_1\beta_1, \alpha_1\beta_2, \ldots, \alpha_1\beta_n), \\
C_1(L \cdot U) &= C_1(A) = (\alpha_1\beta_1, \alpha_2\beta_1, \ldots, \alpha_n\beta_1)
\end{aligned}$$

and

$$(L \cdot U)_{i,j} = \mu_{j-1}(L \cdot U)_{i,j-1} + \Phi(i,j)(L \cdot U)_{i-1,j-1} \\ + \Psi(i,j)(L \cdot U)_{i-1,j} + \Omega(i,j) \tag{7}$$

for $2 \leqslant i, j \leqslant n$.

Let us do the required calculations. First, suppose that $i = 1$. Then

$$(L \cdot U)_{1,j} = \sum_{k=1}^{n} L_{1,k} U_{k,j} = L_{1,1} U_{1,j} = \alpha_1 \beta_j$$

and so $R_1(L \cdot U) = R_1(A) = (\alpha_1\beta_1, \alpha_1\beta_2, \ldots, \alpha_1\beta_n)$.

Similarly, we have

$$(L \cdot U)_{i,1} = \sum_{k=1}^{n} L_{i,k} U_{k,1} = L_{i,1} U_{1,1} = \alpha_i \beta_1$$

and we get $C_1(L \cdot U) = C_1(A) = (\alpha_1\beta_1, \alpha_2\beta_1, \ldots, \alpha_n\beta_1)$.

In what follows, for convenience, we put

$$\Theta(k,j) := (\gamma_{j-1} + \mu_{j-1}\lambda_{k-1}) U_{k,j-1} + (\lambda_{j-1} - \lambda_{k-1}) U_{k,j}. \tag{8}$$

Note that, by Eq. (3) it follows

$$U_{k,j} = \mu_{j-1} U_{k,j-1} + \Theta(k-1, j). \tag{9}$$

Now, suppose that $2 \leqslant i, j \leqslant n$. In this case we have

$$\begin{aligned}
(L \cdot U)_{i,j} &= \sum_{k=1}^{n} L_{i,k} U_{k,j} \\
&= L_{i,1} U_{1,j} + \sum_{k=2}^{n} L_{i,k} U_{k,j} \\
&= L_{i,1} U_{1,j} + \sum_{k=2}^{n} [\delta_{i-1} L_{i-1,k-1} + \Psi(i,k) L_{i-1,k}] U_{k,j} \quad \text{(by (6))} \\
&= L_{i,1} U_{1,j} + \sum_{k=2}^{n} \delta_{i-1} L_{i-1,k-1} U_{k,j} + \sum_{k=2}^{n} \Psi(i,k) L_{i-1,k} U_{k,j} \\
&= L_{i,1} U_{1,j} + \sum_{k=2}^{n} \delta_{i-1} L_{i-1,k-1} [\mu_{j-1} U_{k,j-1} + \Theta(k-1,j)] \\
&\quad + \sum_{k=1}^{n} \Psi(i,k) L_{i-1,k} U_{k,j} - \Psi(i,1) L_{i-1,1} U_{1,j} \quad \text{(by (9))} \\
&= L_{i,1} U_{1,j} + \sum_{k=2}^{n} \mu_{j-1} \delta_{i-1} L_{i-1,k-1} U_{k,j-1} + \sum_{k=2}^{n} \delta_{i-1} L_{i-1,k-1} \Theta(k-1,j) \\
&\quad + \sum_{k=1}^{n} \Psi(i,k) L_{i-1,k} U_{k,j} - \Psi(i,1) L_{i-1,1} U_{1,j}
\end{aligned}$$

$$
\begin{aligned}
&= L_{i,1}U_{1,j} + \sum_{k=2}^{n} \mu_{j-1}\delta_{i-1}L_{i-1,k-1}U_{k,j-1} + \sum_{k=1}^{n} \delta_{i-1}L_{i-1,k}\Theta(k,j) \\
&\quad + \sum_{k=1}^{n} \Psi(i,k)L_{i-1,k}U_{k,j} - \Psi(i,1)L_{i-1,1}U_{1,j} \quad \text{(note that } L_{i-1,n}=0) \\
&= L_{i,1}U_{1,j} + \sum_{k=2}^{n} \mu_{j-1}[L_{i,k} - \Psi(i,k)L_{i-1,k}]U_{k,j-1} \\
&\quad + \sum_{k=1}^{n} \delta_{i-1}L_{i-1,k}\Theta(k,j) \\
&\quad + \sum_{k=1}^{n} \Psi(i,k)L_{i-1,k}U_{k,j} - \Psi(i,1)L_{i-1,1}U_{1,j} \quad \text{(by (6))} \\
&= L_{i,1}U_{1,j} + \sum_{k=1}^{n} \mu_{j-1}[L_{i,k} - \Psi(i,k)L_{i-1,k}]U_{k,j-1} - \mu_{j-1} \\
&\quad \times [L_{i,1} - \Psi(i,1)L_{i-1,1}]U_{1,j-1} \\
&\quad + \sum_{k=1}^{n} \delta_{i-1}L_{i-1,k}[(\gamma_{j-1} + \mu_{j-1}\lambda_{k-1})U_{k,j-1} + (\lambda_{j-1} - \lambda_{k-1})U_{k,j}] \\
&\quad + \sum_{k=1}^{n} \Psi(i,k)L_{i-1,k}U_{k,j} - \Psi(i,1)L_{i-1,1}U_{1,j} \quad \text{(by (8))} \\
&= L_{i,1}U_{1,j} + \sum_{k=1}^{n} \mu_{j-1}L_{i,k}U_{k,j-1} - \sum_{k=1}^{n} \mu_{j-1}\Psi(i,k)L_{i-1,k}U_{k,j-1} \\
&\quad - \mu_{j-1}[L_{i,1} - \Psi(i,1)L_{i-1,1}]U_{1,j-1} \\
&\quad + \sum_{k=1}^{n} \delta_{i-1}(\gamma_{j-1} + \mu_{j-1}\lambda_{k-1})L_{i-1,k}U_{k,j-1} \\
&\quad + \sum_{k=1}^{n} \delta_{i-1}(\lambda_{j-1} - \lambda_{k-1})L_{i-1,k}U_{k,j} \\
&\quad + \sum_{k=1}^{n} \Psi(i,k)L_{i-1,k}U_{k,j} - \Psi(i,1)L_{i-1,1}U_{1,j} \\
&= \mu_{j-1}\sum_{k=1}^{n} L_{i,k}U_{k,j-1} \\
&\quad + \sum_{k=1}^{n}[-\mu_{j-1}\Psi(i,k) + \delta_{i-1}(\gamma_{j-1} + \mu_{j-1}\lambda_{k-1})]L_{i-1,k}U_{k,j-1} \\
&\quad + \sum_{k=1}^{n}[\delta_{i-1}(\lambda_{j-1} - \lambda_{k-1}) + \Psi(i,k)]L_{i-1,k}U_{k,j} \\
&\quad + [L_{i,1} - \Psi(i,1)L_{i-1,1}](U_{1,j} - \mu_{j-1}U_{1,j-1})
\end{aligned}
$$

$$
\begin{aligned}
&= \mu_{j-1} \sum_{k=1}^{n} L_{i,k} U_{k,j-1} + \Phi(i,j) \sum_{k=1}^{n} L_{i-1,k} U_{k,j-1} + \Psi(i,j) \sum_{k=1}^{n} L_{i-1,k} U_{k,j} \\
&\quad + \Omega(i,j) \quad \text{(by (1))} \\
&= \mu_{j-1} (L \cdot U)_{i,j-1} + \Phi(i,j)(L \cdot U)_{i-1,j-1} + \Psi(i,j)(L \cdot U)_{i-1,j} + \Omega(i,j),
\end{aligned}
$$

which is Eq. (7) and the proof is completed. □

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